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pm-Rings and the Prime Ideal Theorem

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Dedicated to Eraldo Giuli on the occasion of his 70th birthday

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ABSTRACT

A commutative ring A with unit is called a *pm-ring* if every prime ideal of A is contained in a unique maximal ideal, and a *Gelfand ring* if $a+b=1$ in A implies that $(1+ar)(1+bs)=0$ for some $r,s\in A$. It was shown earlier, in a somewhat circuitous way involving pointfree topology, that "*pm* implies Gelfand" iff the Prime Ideal Theorem holds. The present note provides an alternative, more direct and entirely ring theoretical proof of a somewhat augmented version of this result.

Recall that the *pm-rings* are the commutative rings with unit in which *every prime ideal is contained in a unique maximal ideal*. These rings, originally introduced by De Marco and Orsatti [4] and subsequently also considered by Mulvey [7] and Johnstone [6], may be viewed as a natural class of abstract rings which capture many features of the concrete rings of continuous real-valued functions on topological spaces. On the other hand, we have the commutative rings with unit in which $a+b=1$ implies $(1+ar)(1+bs)=0$ for some r and s, introduced by Banaschewski [3] as the *Gelfand rings* (although elsewhere that term is also used for the *pm*-rings), and the result, obtained by rather a circuitous route, that every *pm*-ring is a Gelfand ring iff the Prime Ideal Theorem (PIT) holds. It is the present purpose to give a new proof of this which is substantially more direct and based entirely on ring theoretical arguments—which the original proof was not—and at the same time to improve the result somewhat.

Regarding PIT, which asserts that any non-trivial Boolean algebra contains a prime ideal, we recall the basic result of Scott [9] that it implies *the existence of prime ideals in any non-trivial commutative ring with unit* (for a proof, not given in [9] see Rav [8] or Banaschewski [1]). In particular, it then also ensures for any commutative ring A with unit that:

(i) any proper ideal J of A is contained in some prime ideal (consider A/J), and
(ii) for any multiplicative submonoid M of A which misses the zero, there exists a prime ideal disjoint from M (consider the ring of quotients $A[M^{-1}]$).

We begin with a number of auxiliary results.

1. *Every Gelfand ring is a pm-ring.*

To begin with, note that any homomorphic image of a Gelfand ring is a Gelfand ring, as is evident from the definition. Consequently, for any prime ideal P in a Gelfand ring A, the corresponding quotient A/P is a Gelfand *domain* and, as observed already in Banaschewski [3], any such is quite obviously a local ring (for any element c, c or $1-c$ is invertible). Hence A/P has a unique maximal ideal, given by all its non-invertible elements, which then determines a unique maximal ideal in A above P.

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2. *If PIT holds then every pm-ring is Gelfand.*

Let A be any *pm*-ring, $a+b=1$ for some $a,b\in A$, and suppose that $(1+ar)(1+bs)\neq 0$ for all $r,s\in A$. Now,

$$M=\{(1+ar)(1+bs)\mid r,s\in A\}$$

is evidently a multiplicative submonoid of A since

$$(1+cx)(1+cy)=1+cz \quad \text{for } z=x+y+cxy,$$

and given that $0\notin M$ A has a prime ideal P disjoint from M by PIT. Next, $P+Aa\subset A$ since $P+Aa=A$ implies $x+ay=1$ for some $x\in P$ and $y\in A$ and hence $x=1-ay\in P\cap M$, a contradiction. Similarly, $P+Ab\subset A$ and consequently A has prime ideals $Q\supseteq P+Aa$ and $S\supseteq P+Ab$, again by PIT. Further, since A is a *pm*-ring, there exist maximal ideals $U\supseteq Q$ and $V\supseteq S$ in A, but then also $U,V\supseteq P$ and therefore $U=V$ so that $1=a+b\in U$, a contradiction.

In the following, a ring A will be called *idempotent-generated* if it is commutative with unit and generated, as a ring, by its idempotents so that every element of A is just an integral linear combination of idempotents.

3. *No idempotent-generated ring with torsion-free additive group is Gelfand.*

Consider any ring A of the kind in question and suppose it is Gelfand. Then, since $3-2=1$ in A, there exist $r,s\in A$ such that $(1+3r)(1-2s)=0$. Now let B be the subalgebra of the Boolean algebra $Idp\,A$ of idempotents of A generated by any finitely many idempotents which allow to express r and s as integral linear combinations. Then B is finite and consequently atomic, and r and s are integral linear combinations of atoms of B. As a result, if $u\in B$ is any atom then $ru=ku$ and $su=\ell u$ with suitable $k,\ell\in\mathbf{Z}$ and hence

$$(1+3k)(1-2\ell)u=0;$$

finally, by the properties of A, it follows that $(1+3k)(1-2\ell)=0$ in $\mathbf{Z}$, implying that $1=-3k$ or $1=2\ell$, a contradiction.

We note in passing that the above hypothesis of torsion-freeness cannot be dropped: any Boolean ring is Gelfand since $a+b=1$ implies $ab=0$ and hence $(1+a)(1+b)=0$.

Now we have the desired:

Proposition. *The pm-rings are exactly the Gelfand rings iff the Prime Ideal Theorem holds.*

Proof. ($\Rightarrow$) If every *pm*-ring is Gelfand then any idempotent-generated ring with torsion-free additive group must have a prime ideal because otherwise it would be (vacuously) *pm* and therefore Gelfand, contradicting 3. On the other hand, it is a familiar fact that any Boolean algebra can be embedded into the Boolean algebra $Idp\,A$ of idempotents of some commutative ring with unit whose additive group is torsion-free. Of course, the familiar way of seeing this, using the Stone Representation Theorem to turn the elements of the given Boolean algebra into the characteristic functions of the open-closed sets of the representation space, is not available here because that theorem is equivalent to the Prime Ideal Theorem. However, there is an alternative, purely algebraic approach which avoids this problem, based on Foster's notion of Boolean powers (see [5]), here specifically employed in the form of the Boolean power $\mathbf{Z}[B]$ of the ring $\mathbf{Z}$ of integers relative to the given Boolean algebra B. We briefly recall the details for the sake of completeness. The *elements* of $\mathbf{Z}[B]$ are the maps $a,b,\ldots:\mathbf{Z}\to B$ such that:

(i) $a(m)=0$ for all but finitely many $m\in\mathbf{Z}$,
(ii) $a(k)\wedge a(\ell)=0$ whenever $k\neq\ell$, and
(iii) $\bigvee\{a(m)\mid m\in\mathbf{Z}\}=e$, the unit of B

(the latter evidently being a finite join by (i)), while its *operations* are determined by the operations of $\mathbf{Z}$ as follows:

$$(a\dot{+}b)(m)=\bigvee\{a(k)\wedge b(\ell)\mid k\dot{+}\ell=m\},$$
$$(-a)(m)=a(-m),$$
$$\mathbf{0}(0)=e=\mathbf{1}(1).$$

By the general properties of Boolean powers, $\mathbf{Z}[B]$ is then a commutative ring with unit $\mathbf{1}$ (and zero $\mathbf{0}$), and its additive group is easily checked to be torsionfree. Further, for any $s\in B$, $u_s\in\mathbf{Z}[B]$ such that

$$u_s(1)=s \quad \text{and} \quad u_s(0)={\sim}s, \quad \text{the complement of } s,$$

is idempotent and the map $s\mapsto u_s$ is a Boolean homomorphism $B\to Idp(\mathbf{Z}[B])$, evidently one-to-one. Now, the subring A of $\mathbf{Z}[B]$ generated by these u_s, $s\in B$, has a prime ideal P, as stated, and $\{s\in B\mid u_s\in P\}$ is then obviously a prime ideal of B, proving PIT. It might be added here that, in actual fact, $A = \mathbf{Z}[B]$—but this is certainly not relevant for the present argument.

($\Leftarrow$) Clear from 1 and 2. $\square$

Remark. It should be noted that the above Boolean power $\mathbf{Z}[B]$ can also be described as the ring of continuous integer-valued functions on the frame $\mathfrak{J}B$ of ideals of B.

In closing, we briefly turn to a related situation, again involving the Gelfand rings but now comparing them with the commutative rings A with unit such that:

For any distinct maximal ideals P and Q of A, there exist $r \notin P$ and $s \notin Q$ in A such that $rs = 0$.

These rings were introduced by Mulvey [7], and then also considered by Johnstone [6], as an alternative form of the *pm*-rings: they turn out to be the same as the latter if the Axiom of Choice (AC) is assumed and are particularly suitable for the purposes involved in [7]. Subsequently, they were also dealt with by Banaschewski [3], where they were called *weakly Gelfand*, and their relation to the Gelfand rings was clarified by the following result:

The weakly Gelfand rings are exactly the same as the Gelfand rings iff the Axiom of Choice holds.

Similar to the previous case, the proofs of this given in [3] involves considerations in pointfree topology but again there is a more direct, entirely ring theoretic argument which proceeds as follows.

First, that Gelfand implies weakly Gelfand is straightforward and independent of any choice principle (a fact which motivated the terminology). Secondly, deriving the reverse implication from AC works in much the same way as the proof of 2 but here one uses AC to obtain maximal ideals $U \supseteq P + Aa$ and $V \supseteq P + Ab$, obviously distinct, for which $rs = 0$ with $r \notin U$ and $s \notin V$ then readily leads to the desired contradiction since $r \in P$ or $s \in P$. Finally, to obtain AC from the hypothesis that weakly Gelfand is the same as Gelfand a radically new line of thought is required: based on the considerations involved in Banaschewski [2], one shows that $\neg$AC leads to the existence of a weakly Gelfand ring which fails to be Gelfand.

For this, recall from [2] that, for any non-void set E with a partition $\mathfrak{A}$ into non-void subsets, there exists a ring of fractions A of the polynomial ring $\mathbf{Q}[E]$ with E as its set of indeterminates such that the maximal ideals of A correspond exactly to the subsets $S \subseteq E$ which meet each $X \in \mathfrak{A}$ in exactly one element. Hence $\neg$AC implies there is such a ring A without any maximal ideals which in turn makes it (vacuously) weakly Gelfand. On the other hand, since A is an integral domain it is Gelfand iff it is a local ring, as already noted earlier, but then it has a maximal ideal—which it does not.

Finally, it should be noted that the present ring A, again since it is an integral domain, also fails to be a *pm*-ring so that, in addition, we can conclude:

The weakly Gelfand rings are exactly the pm-rings iff the Axiom of Choice holds.

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